Mathematical Derivation of Andre Ampère's Law Using Special Relativity Theory

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Abstract: The Andre Ampère's Law is a mathematical equation that describes the force between two current carrying wire segments called current elements. It was created by Ampère from a series of experiments in about 1822. Andre Ampère was a French physicist and mathematician. The purpose of this paper is to show the mathematical connection between his experimentally deduced law and Einstein's Special Relativity (SR), which was formulated almost a hundred years later. This connection, with experiment, will verify Andre Ampère's Law as the correct one from the many like laws that exist. In addition, Andre Ampère's experiment will add an experimental verification of SR at slow relative velocities. This mathematical derivation is performed using four-vector math from SR and results are presented in three dimensions. Experimental verification of the law is not presented in this paper because the fact the law can be correctly mathematically derived is a fact that needs to be assimilated by the reader and is enough for this paper. Experimental verification is later.

Key Words: SR, Lorentz transform, four vector, three vector, electric field (e-field), relative moving charges, magnetic force, Coulomb force, steady-state, current element.

Introduction.

This paper is important just to get the field of Electrodynamics back on a better track than the one developed by Maxwell and Faraday back in the mid19th century. Andre Ampère determined his law from a set of very careful experiments and was highly praised later by James Clerk Maxwell.

Several physicists have written similar laws. They are Weber, Grassmann, Maxwell, Neumann, etc. The laws that Maxwell and Grassmann have written are popular ones in the current physics literature. Weber's law is very similar to Andre Ampère's Law, and as such deserves a lot of credit. It is currently promoted by Andre Assis of Brazil.

Algebra and Vector Analysis are the mathematical tools employed in this paper. Analysis is done in 3 dimensions contrasted to 2 dimensions such as having two parallel current elements.

The current element, vectors, and some definitions.

A current element is a short piece of a conductor of magnitude length *ds* and is carrying a current of magnitude *I.* It possesses an electron or proton line charge density σ. The electrons are moving with velocity v and creating the current I. The relationship between the moving electrons and the current I is given by the following formula: $\sigma ds = -Ids / v$. By convention the direction of the current is opposite the direction of the velocity of the electrons. We set the velocity vector $\mathbf{v}$ of the electrons in

the same direction as the direction of the position vector position vector $\mathbf{ds}$. Remember that the vectors $\mathbf{ds}$, $\mathbf{v}$, and $\mathbf{r}$ have magnitudes and components in the x, y, and z axis of a 3D coordinate system.
These relationships are shown in Figure 1:.

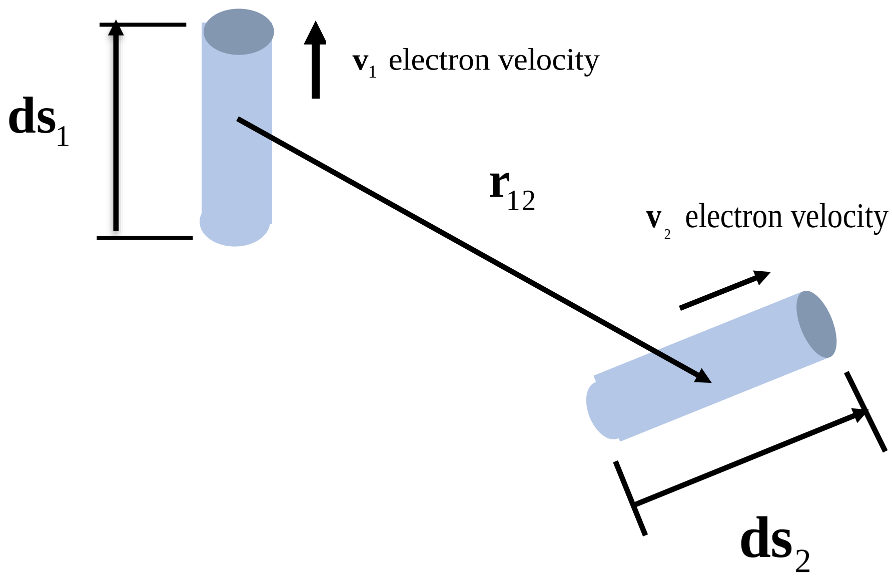

Figure 1. Vector relationships of the current elements in this derivation.

Andre Ampère's Law.

This law allows one to calculate the force between two current elements as depicted in Figure 1. This force is directed along the position vector $\mathbf{r}_{12}$. It can be repulsive or attractive, depending on the direction of the vectors. This law is a steady state law which means the currents in the current elements are constant. The law as determined by Andre Ampère is given by the following formula[1]:

$$d^2\mathbf{f_{12}} = -\hat{\mathbf{r}}_{12}\frac{\mu_o I_1 I_2 ds_1 ds_2}{4\pi r_{12}{}^2}\left[2\sin(\alpha_1)\sin(\alpha_2)\cos(\eta) - \cos(\alpha_1)\cos(\alpha_2)\right] \quad (1)$$

The angle terms on the right side of Eq. (1) were determined experimentally by Andre Ampère. This author will not define the angle terms in Eq. (1) since they can be expressed equally in vector notation by the following formula[1]:

$$d^2\mathbf{f_{12}} = -\hat{\mathbf{r}}_{12}\frac{\mu_o I_1 I_2 ds_1 ds_2}{4\pi r_{12}{}^2}\left(2d\hat{\mathbf{s}}_1 \bullet d\hat{\mathbf{s}}_2 - 3(d\hat{\mathbf{s}}_1 \bullet \hat{\mathbf{r}}_{12})(d\hat{\mathbf{s}}_2 \bullet \hat{\mathbf{r}}_{12})\right) \quad (2)$$

Eq. (1) and Eq. (2) are mathematically equivalent. Eq. (2) is the defined target of this derivation.

Basic considerations.

1. A uniform electric field exists about a stationary charge of an electron or proton. The charge is assumed to be a point charge and does not have a finite size.
2. The field of the point charge is assumed to pervade all space and falls off in intensity at $1/r^2$.
3. Velocities are relative between interacting particles.
4. The electric field of a moving charge pervades all the space of its inertial frame, thus having instant reaction with a charge in contact with the field. Acceleration of a charge creates a new velocity that changes the electric field that spreads out over the new inertial frame at the speed of light.
5. The charge q is invariant from one inertial frame to another.
6. A positive sign on the overall magnetic force represents repulsion while a negative sign represents attraction.
7. A negative sign must be entered into the equations for negative charges such as electrons. A positive sign must be entered into the equations for positive charges such as protons. This makes the direction of the force appear correctly as in 6 above.

Special Relativity Theory (SRT) and formalism employed.

1. Basic Postulates:
   a. Relativity Principle(RP): "all inertial frames are totally equivalent for the performance of all physical experiments".
   b. "Light travels rectilinearly with constant speed c in vacuum in every inertial frame".
2. Lorentz Transformation is used.
3. Gamma factor: $\gamma = 1/\sqrt{1 - v^2/c^2}$
4. Three-vectors: a(a_1, a_2, a_3)
5. Four-vectors: V(V_1, V_2, V_3)
6. Four-vector force formula: $\mathbf{F} = \gamma(v)\left(\mathbf{f}, \frac{\mathbf{f} \bullet \mathbf{v}}{c^2}\right)$
7. The four-vector force formula is good for transforming force between internal frames and creating force laws.
8. The derivation of the four-vector force formula and how the individual components are transformed between relative moving frames is found in this book [2] on Special Relativity.

Starting point for the four-force SRT transformation of the force between relatively moving charges.

The Lorentz Force formula (full law) is the starting point for relativists as depicted in literature:

$$\mathbf{f} = q\left(\mathbf{e} + \frac{\mathbf{v}\times\mathbf{h}}{c^2}\right) \tag{3}$$

The relativists transformation results are two formulas:

$$\mathbf{h} = \frac{1}{c}\mathbf{v}\times\mathbf{e} \quad , \quad \mathbf{e} = \frac{q\mathbf{r}}{\gamma^2 r^3\left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{\frac{3}{2}}}$$

The author contends the full Lorentz force law is not appropriate for the four-force transformation between relatively moving isolated charges. The regular Lorentz Force Law may be applicable in a cathode ray tube or accelerators where the source of the magnetic field is separate from the "magnetic field" created by the moving charge. The author expects that the "magnetic field" will fall out of the four-force transformation of the first term of the Lorentz Force formula. So, the author takes the first term and does a four-force transformation on it: $\mathbf{f} = q\mathbf{e}$. The results are:

$$\mathbf{e} = \frac{q\mathbf{r}}{\gamma^2 r^3\left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{\frac{3}{2}}} \tag{4}$$

The electric field of a stationary charge is defined by $\mathbf{e} = \mathrm{k}q\mathbf{r}/r^3$ in RMKS units where k=1/4$\pi\varepsilon_o$. ε_o is the permittivity of free space. So in RMKS units we multiply the numerator of Eq. (3) by k.

Theta in Eq. (4) is the angle between $\mathbf{r}_{12}$ and $\mathbf{v}$. See Figure 2.

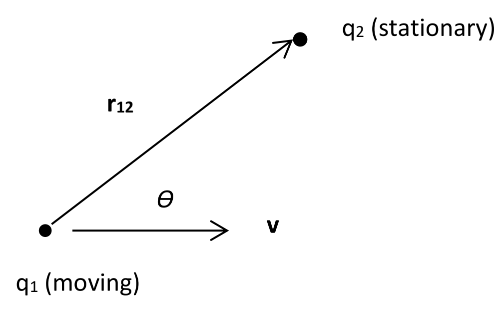

Figure 2. Relation between the vector terms in Eq. (3) as applied to two charges (q_1 is q in Eq.(4)). θ is angle between $\mathbf{r_{12}}$ and $\mathbf{v}$.

It is helpful to visualize the **e**-field of q_1 the moving charge, in Eq. (4) and compare it to an **e**-field of a stationary charge. This is shown in Figure 3. **e**-fields are generally represented by radial lines emitting from a charge. The number of **e**-field flux lines per area is the magnitude or strength of the **e**-field. Multiply the **e**-field at a point in space by a charge at that point in space, then you have the force on that charge.

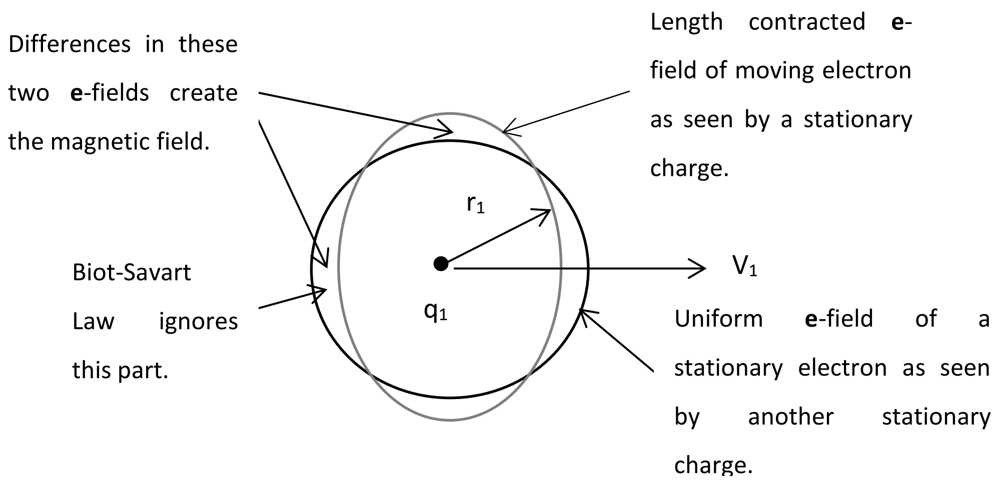

Figure 3. How special relativity (SR) creates the magnet field of moving charges. This shows a cross section of the effect on the e-field of a moving charge represented by Eq. (4).

The magnetic force law between slow relative moving isolated charges.

We are now ready to derive magnet force between relative slow moving isolated charges.

An equation very similar to the Lorentz Force Law is derived from Eq. (4) using the Binomial Series and eliminating higher order terms of v^2/c^2 (this derivation is shown in Appendix A):

$$\mathbf{f}_{12} = q_2\left[\frac{kq_1}{r^2} + \frac{kq_1}{r_{12}{}^2}\frac{v^2}{c^2}\left(0.5\text{-}1.5\cos^2\theta\right)\right]\hat{\mathbf{r}}_{12} \tag{5}$$

Eq. (5) describes the total force between two relative slow moving isolated charges. The first term describes a uniform **e**-field (Coulomb field) as in the Lorentz Force Law and the second term describes magnetic effects as does the second term in the Lorentz force law. Also, the relative velocity v between the two charges is clearly defined, as in Figure 2, and not obscure as in the Lorentz Force Law. The force in Eq. (4) acts along $\mathbf{r}_{12}$ and does not violate Newton's Third Law as the Lorentz Force Law does [3]. Eq. (5) could be written as:

$$\mathbf{f}_{12} = q_2\left(e_1 + e_{m1}\right)\hat{\mathbf{r}}_{12} \tag{6}$$

where e_{m1} is the new definition of the magnetic field created by the moving charge q_1:

$$\mathbf{e_{m1}} = \frac{kq_1}{r_1^2}\frac{v^2}{c^2}\left(0.5 - 1.5\cos^2\theta\right)\hat{\mathbf{r}}_1 \tag{7}$$

The definition of a field, electric or magnetic, is force per unit charge. Consequently, multiplying $\mathbf{e}_{\mathbf{m}1}$ by q_2 gives magnetic force between relatively slow-moving isolated charges.

The magnetic force Law between a stationary current element and a stationary charge.

This steady state law Eq. (8) is derived from Eq. (4) in the same manner as Eq. (7) with the exception that q_1 in Eq. (4) is replaced with $\gamma\sigma_1 ds_1$, where σ_1 is line charge density and $\sigma_1 ds_1$ is the current element charge (this derivation is shown in Appendix B):

$$d\mathbf{f}_{12} = \frac{kq_2\sigma_1 ds_1}{r^2}\frac{v^2}{c^2}\hat{\mathbf{r}}_{12}\left(1\text{-}1.5\cos^2\theta\right) \tag{8}$$

The γ in $\gamma\sigma_1 ds_1$ expresses length contraction of the distance between the moving charges in the current element as seen by the test charge q_2 thereby increasing the effective charge density it sees. Notice the subtle differences between Eq. (7) and Eq. (8). The "0.5" in Eq. (7) is replaced by "1.0" in Eq. (8) and the moving charge q_1 is replaced by the current element charge $\sigma_1 ds_1$. Also, notice that Eq. (8) describes a force between a stationary charge and a "magnetic field" e_{m1} of a stationary current element. There is no need to have q_2 moving with respect to the current element as would be required by the Lorentz Force Law Eq. (3). This force has been reported in the works of Cooper [4] and Spencer [5]. Eq. (8) is very similar to Gauss's Law, Eq. (9), (with the Coulomb term removed) derived in about 1835 [6]:

$$\mathbf{f}_{12} = \frac{kq_1q_2}{r_{12}^2}\frac{v^2}{c^2}\hat{\mathbf{r}}_{12}\left(1\text{-}1.5\cos^2\theta\right) \tag{9}$$

One or both of the charges in Gauss's Law should be a current element charge. Apparently, Gauss, a mathematician, "back engineered" the classical Ampère's Law to arrive at Eq.(9). The electron and special relativity had not been discovered when he derived this law. Gauss's Law can denote a magnet force between two current elements. It is important to recognize that Eq. (8) and Eq. (9) describes the magnetic force between a stationary charge q_2 or a stationary $\sigma_2 ds_2$, and a current element charge $\sigma_1 ds_1$. In Eq. (8) the velocity term v is the relative velocity between the positive ion lattice, which is stationary relative to q_2, and the moving electron lattice creating the current. When the equation is applied to the moving electron lattice of another current element the velocity term is the relative velocity of the electrons in that lattice and the moving electron lattice of the other current element.

Eq. (8) is employed to derive the classical Ampère's Law Eq. (2). Cross combination force relationships of Eq. (8), three altogether, are added together to form the classical Ampère's Law as describes below. This is not a trivial derivation and is achieved here for the general 3-dimensional case using vector analysis. Once Ampère's Law has been derived, the connection of special relativity to electrodynamics is confirmed. Peter Graneau's work on "The Ampère-Neumann Electrodynamics of Metals" [7] is then applicable.

The magnetic force law between current elements, classical Andre Ampère's Law.

The test charge q_2 in Eq. (8) may be replaced by $\sigma_2 ds_2$ of another wire current element with charge line density σ_2. Then the two $\sigma ds's$ are replaced by two $Ids/v's$, the v corresponding to the moving charges in each of the two current elements. The positive ion lattices of the two stationary current elements are stationary with respect to each other. Their Coulomb repulsion has already been subtracted out in Eq. (8) and this force does not need to be added. Then applying Eq. (8) three times to the cross combination of charges in the two current elements and adding the forces one arrives at the classical Ampère's Law.

Let $\mathbf{A} = \frac{k\sigma_2 ds_2 \sigma_1 ds_1}{r^2 c^2}\hat{\mathbf{r}}_{12}$ and $\cos^2(\theta_1) = (\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})^2$ and $\cos^2(\theta_2) = (\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12})^2$

First charge combination of proton lattice in ds_2 with electron lattice in ds_1:

$$d^2\mathbf{f}_{12} = -\mathbf{A}[v_1^2 - v_1^2 1.5(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})^2]$$

Second charge combination of proton lattice in ds_1 with electron lattice in ds_2:

$$d^2\mathbf{f}_{21} = -\mathbf{A}[v_2^2 - 1.5v_2^2(\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{21})^2]$$

Third charge combination of electron lattice in ds_1 with electron lattice in ds_2:

Determining this is more involved because one must determine the relative velocity and the angle between the relative velocity vector **v** and the position vector **r.** This is not impossible. First, we set up a vector diagram describing the situation and then employ the law of cosines; see Figure 4:

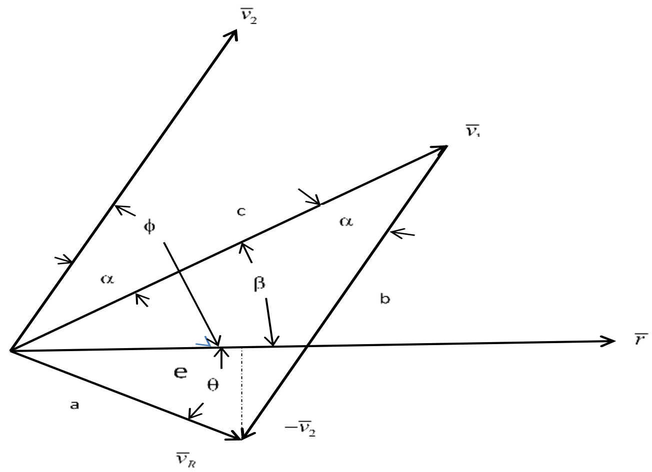

Figure 4. Vector diagram for determining relative velocity vector between two velocity vectors and also the $\cos(\theta)$ term of Eq. (8). This is a very important diagram used for deriving Andre Ampère's Law.

Perform the following mathematical steps to determine the $\cos(\theta)$ and the relative velocity of the electron lattices v_R in Eq. (8), refer to Figure 4:

Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ so $v_R^2 = v_1^2 + v_2^2 - 2v_1v_2(\hat{\mathbf{v}}_1 \bullet \hat{\mathbf{v}}_2)$

Where $\mathbf{v}_1 = v_1\hat{\mathbf{v}}_1$ $\mathbf{v}_2 = v_2\hat{\mathbf{v}}_2$ $\cos(\alpha) = \hat{\mathbf{v}}_1 \bullet \hat{\mathbf{v}}_2 = \hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{ds}}_2$

Also $e = v_1\cos(\beta) - v_2\cos(\phi) = v_1(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12}) - v_2(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12})$

and

$$\cos^2(\theta) = \frac{e^2}{v_R^2} = \frac{\left[v_1(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12}) - v_2(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12})\right]^2}{v_R^2} = \frac{v_1^2(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12})^2 - 2v_1v_2(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12})(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12}) + v_2^2(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12})^2}{v_R^2}$$

With $\mathbf{A} = \frac{k\sigma_2 ds_2 \sigma_1 ds_1}{r^2 c}\hat{\mathbf{r}}_{12}$

$$d^2\mathbf{f}_{ee} = \frac{k\sigma_2 ds_2 \sigma_1 ds_1}{r^2}\frac{v_R^2}{c^2}\hat{\mathbf{r}}_{12}\left(1\text{-}1.5\cos^2\theta\right) = \mathbf{A}(v_R^2 - v_R^2 1.5\cos^2\theta)$$

Adding magnetic forces:

$$d^2\mathbf{f}_{\text{total}} = d^2\mathbf{f}_{12} + d^2\mathbf{f}_{21} + d^2\mathbf{f}_{\text{ee}}$$

$$d^2\mathbf{f}_{\text{total}} = -\mathbf{A}[v_1^2 - v_1^2 1.5(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12})^2] - \mathbf{A}[v_2^2 - 1.5v_2^2(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{21})^2]$$

$$+\mathbf{A}(v_1^2 + v_2^2 - 2v_1v_2(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{ds}}_2) - A1.5[v_1^2(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12})^2 - 2v_1v_2(\hat{\mathbf{ds}}_1 \bullet \hat{\mathbf{r}}_{12})(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12}) + v_2^2(\hat{\mathbf{ds}}_2 \bullet \hat{\mathbf{r}}_{12})^2])$$

$$d^2\mathbf{f}_{total} / \mathbf{A} = -v_1^2 + v_1^2 1.5(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})^2 - v_2^2 + 1.5v_2^2(\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{21})^2$$
$$+v_1^2 + v_2^2 - 2v_1v_2(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{ds}}_\mathbf{2}) - 1.5v_1^2(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})^2 + 3v_1v_2(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})(\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12}) - 1.5v_2^2(\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12})^2$$
$$= -2v_1v_2(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{ds}}_\mathbf{2}) + 3v_1v_2(\hat{\mathbf{ds}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})(\hat{\mathbf{ds}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12})$$

Andre Ampère's Law (RMKS units):

Substituting back in A and replacing $\sigma_1 ds_1$ with $I_1 ds_1 / v_1$ and $\sigma_2 ds_2$ with $I_2 ds_2 / v_2$:

$$d^2\mathbf{f}_\mathbf{12} = -\hat{\mathbf{r}}_{12} \frac{kI_1I_2ds_1ds_2}{c^2{r_{12}}^2}\left(2\mathbf{d}\hat{\mathbf{s}}_\mathbf{1} \bullet \mathbf{d}\hat{\mathbf{s}}_\mathbf{2} - 3(\mathbf{d}\hat{\mathbf{s}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})(\mathbf{d}\hat{\mathbf{s}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12})\right) \quad (10)$$

With $k = \frac{1}{4\pi\varepsilon_o} = \frac{\mu_o c^2}{4\pi}$, where μ_o is permeability of free space:

$$d^2\mathbf{f}_\mathbf{12} = -\hat{\mathbf{r}}_{12} \frac{\mu_o I_1I_2ds_1ds_2}{4\pi{r_{12}}^2}\left(2\mathbf{d}\hat{\mathbf{s}}_\mathbf{1} \bullet \mathbf{d}\hat{\mathbf{s}}_\mathbf{2} - 3(\mathbf{d}\hat{\mathbf{s}}_\mathbf{1} \bullet \hat{\mathbf{r}}_\mathbf{12})(\mathbf{d}\hat{\mathbf{s}}_\mathbf{2} \bullet \hat{\mathbf{r}}_\mathbf{12})\right) \quad \text{Q.E.D.} \quad (11)$$

Conclusion.

Eq. (11) was experimentally verified by Andre Ampère in about 1822. It now has the backing of SR. This law and the equations leading to its derivation are also legitimate formulas and should not be ignored by physicists and engineers. (The author has shown in a paper[8] how Eq. (11) can be used in a computer to calculate values by using a finite length of the current element ds. This avoids long computer calculations having ds to be almost infinitely small.)

Acknowledgement

I want to acknowledge A. K. T. Assis, and J.P.M. C. Chaib for publishing the English translation of Ampère's masterpiece. It can be found here:
http://www.ifi.unicamp.br/~assis/Amperes-Electrodynamics.pdf

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Appendix A

Derivation of the electric and magnetic force between slow relative moving isolated charges.

Refer to Eq. (3) and Figure 2 in the paper's text. Multiply Eq. (3) by kq_2 and we get a formula (in RMKS units) Eq. (1a) for force between two moving isolated charges that should be good up to the speed of light. However, for Andre Ampère's Law, the relative velocities are slow and we need an approximation of Eq.(1a).

$$\mathbf{f}_{12} = \frac{kq_2 q_1 \mathbf{r}_{12}}{\gamma^2 r^3 \left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{\frac{3}{2}}} \tag{1a}$$

Let $\mathbf{a} = \dfrac{kq_1 q_2 \mathbf{r}_{12}}{r^3}$, $d = \dfrac{1}{\gamma^2} = \left(1-\dfrac{v^2}{c^2}\right)$, $g = (1-\dfrac{v^2}{c^2}\sin^2\theta)^{-\frac{1}{2}}$, then $\mathbf{f}_{12} = \mathbf{a}dg^3$

Binomial Series (Standard Form):

$$\left(1\pm x\right)^n = 1 \pm \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots \quad x^2 < 1, n \neq 0,1,2\ldots$$

Applying to g:

$$x = \frac{v^2}{c^2}\sin^2\theta \;\; , \;\; n = -\frac{1}{2} \;\; , \;\; g = (1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots$$

Eliminating higher orders of magnitude greater than $\dfrac{v^2}{c^2}$, $g = 1 + \dfrac{1}{2}x$

Let $z = \dfrac{1}{2}x = \dfrac{v^2}{2c^2}\sin^2\theta$, $z^3 = (1+z)(1+z)1+z) = 1 + 2z + z^2 + z + 2z^2 + z^3$

Eliminating higher orders of magnitude greater than $\dfrac{v^2}{c^2}$, $g^3 \simeq 1 + 3z$

Let $y=\dfrac{v^2}{c^2}$, then $dg^3=(1-y)(1+3z)=1+3z-y-3zy$

Eliminating higher orders of magnitude greater than $\dfrac{v^2}{c^2}$ and substituting values for y and z:

$$dg^3=1+1.5\frac{v^2}{c^2}\sin^2\theta-\frac{v^2}{c^2}=1+\frac{v^2}{c^2}\left(-1+1.5\sin^2\theta\right)$$

$$=1+\frac{v^2}{c^2}\left(-1+1.5\left(1-\cos^2\theta\right)\right)=1+\frac{v^2}{c^2}\left(0.5-1.5\cos^2\theta\right)$$

Applying **a**: $\mathbf{f}_{12}=\dfrac{kq_1q_2\mathbf{r}_{12}}{r^3}\left(1+\dfrac{v^2}{c^2}\left(0.5-1.5\cos^2\theta\right)\right)$

Appendix B

Derivation of the electric and magnetic force between a stationary current element and a statioary charge.

Refer to Eq. (3) and Figure 2 in the paper's text. Multiply Eq. (3) by $k\gamma q_1$ and we get a formula (in RMKS ubits) Eq. (1b) for force between a stationary current element and a stationary charge. Remember that q_1 can represent an isolated charge like an electron or a group of isolated charges like an electron lattice in a current element. The latter is the case for Eq. (1b).The insertion of γ in the numerator of Eq. (1b) represents length contraction of the electron lattice of q_1 as seen by the stationary charge q_2 .This inseration will not violet any SRT rules. q_1 will later be replaced by $\sigma_1 ds_1$ where σ_1 is the line charge density of current element 1. For Andre Ampère's Law, the relative velocities are slow and we need an approximation of Eq.(1b).

$$\mathbf{f}_{12}=\frac{k\gamma q_2q_1\mathbf{r}_{12}}{\gamma^2r^3\left[1-\left(v^2/c^2\right)\sin^2\theta\right]^{\frac{3}{2}}} \tag{1b}$$

Let $\mathbf{a}=\dfrac{kq_1q_2\mathbf{r}_{12}}{r^3}$, $d=\dfrac{1}{\gamma}=\left(1-\dfrac{v^2}{c^2}\right)^{\frac{1}{2}}$, $g=(1-\dfrac{v^2}{c^2}\sin^2\theta)^{-\frac{1}{2}}$, Then $\mathbf{f}=\mathbf{a}dg^3$

Binomial Series (Standard Form):

$$\left(1\pm x\right)^n=1\pm\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\ldots \qquad x^2<1, n\neq 0,1,2\ldots$$

Applying to g:

$$x=\frac{v^2}{c^2}\sin^2\theta \quad , \quad n=-\frac{1}{2} \quad , \quad g=(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+...$$

Eliminating higher orders of magnitude greater than $\frac{v^2}{c^2}$, $\quad g=1+\frac{1}{2}x$

Let $z=\frac{1}{2}x \quad , \quad g^3=(1+z)(1+z)1+z)=1+2z+z^2+z+2z^2+z^3$

Eliminating higher orders of magnitude greater than $\frac{v^2}{c^2}$, $\quad g^3\simeq 1+3z$

For approximating *d* use the Binomial Series letting $x=\frac{v^2}{c^2}$ and $n=\frac{1}{2}$

$$d=\left(1-x\right)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1}{8}x^2+...$$

Eliminating higher orders of magnitude greater than $\frac{v^2}{c^2}$, $\quad d\simeq 1-\frac{1}{2}\frac{v^2}{c^2}$

Let $y=\frac{1}{2}\frac{v^2}{c^2}$, then $dg^3=\left(1-y\right)\left(1+3z\right)=1+3z-y+3zy$

Eliminating higher orders of magnitude greater than $\frac{v^2}{c^2}$ and substituting values for y and z:

$$dg^3=1+1.5\frac{v^2}{c^2}\sin^2\theta-\frac{1}{2}\frac{v^2}{c^2}=1+\frac{v^2}{c^2}\left(-0.5+1.5\sin^2\theta\right)$$

$$=1+\frac{v^2}{c^2}\left(-0.5+1.5\left(1-\cos^2\theta\right)\right)=1+\frac{v^2}{c^2}\left(1-1.5\cos^2\theta\right)$$

Applying **a**: $\mathbf{f}=\frac{kq_1q_2\mathbf{r}_{12}}{r^3}\left(1+\frac{v^2}{c^2}\left(1-1.5\cos^2\theta\right)\right)$

Notes:

1. Formulas updated 10/7/22. JK